

Faster FPT Algorithm for Graph Isomorphism Parameterized by Eigenvalue Multiplicity

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Abstract

We give a $O^*(k^{O(k)})$ time isomorphism testing algorithm for graphs of eigenvalue multiplicity bounded by k which improves on the previous best running time bound of $O^*(2^{O(k^2/\log k)})$ [EP97a].¹

1 Introduction

Two simple undirected graphs $X = (V, E)$ and $X' = (V', E')$ are said to be *isomorphic* if there is a bijection $\varphi : V \rightarrow V'$ such that for all pairs $\{u, v\} \in \binom{V}{2}$, $\{u, v\} \in E$ if and only if $\{\varphi(u), \varphi(v)\} \in E'$. Given two graphs X and X' as input the decision problem *Graph Isomorphism* asks whether X is isomorphic to X' . An outstanding open problem in the field of algorithms and complexity is whether the Graph Isomorphism problem has a polynomial-time algorithm. The asymptotically fastest known algorithm for Graph Isomorphism has worst-case running time $2^{O(\sqrt{n \lg n})}$ on n -vertex graphs [BL83]. On the other hand, the problem is unlikely to be NP-complete as it is in $\text{NP} \cap \text{coAM}$ [BHZ87].

However, efficient algorithms for Graph Isomorphism have been discovered over the years for various interesting subclasses of graphs, like, for example, bounded degree graphs [Luks80], bounded genus graphs [Mil80, GM12], bounded eigenvalue multiplicity graphs [BGM82, EP97a].

The focus of the present paper is Graph Isomorphism for bounded eigenvalue multiplicity graphs. This was first studied by Babai et al [BGM82] who gave an $n^{O(k)}$ time algorithm for it. There is also an NC algorithm² for the problem for constant k due to Babai [Bab86]. Using an approach based on cellular algebras and some nontrivial group theory, Evdokimov and Ponomarenko [EP97a] gave an $O^*(2^{O(k^2/\log k)})$ algorithm for it. This puts the problem in FPT, which is the class of *fixed parameter tractable* problems. The parameter in question here is the bound k on the eigenvalue multiplicity of the input graphs.

¹Throughout the paper, we use the $O^*(\cdot)$ notation to suppress multiplicative factors that are polynomial in input size.

²NC denotes the class of problems that can be solved in the parallel-RAM model in polylogarithmic time using polynomially many processors.

In this paper we obtain a $O^*(k^{O(k)})$ time isomorphism algorithm for graphs of eigenvalue multiplicity bounded by k . We follow a relatively simple geometric approach to the problem using integer lattices. Recently, we obtained an $O^*(k^{O(k)})$ time algorithm for *Point Set Congruence* (abbreviated GGI) in \mathbb{Q}^k in the ℓ_2 metric [AR14]. Our algorithm is based on a lattice isomorphism algorithm of running time $O^*(k^{O(k)})$, due to Haviv and Regev [HR14]. They design an $O^*(n^{O(n)})$ time algorithm for checking if two integer lattices in \mathbb{R}^n are isomorphic under an orthogonal transformation. In [AR14] we adapt their technique to solve the Point Set Congruence problem, GGI, in $O^*(k^{O(k)})$ time.

Now, in this paper, building on our previous algorithm for GGI [AR14], combined with some permutation group algorithms, we first give an $O^*(k^{O(k)})$ time algorithm for a suitable *geometric automorphism* problem, defined in Section 4. It turns out that the bounded eigenvalue multiplicity Graph Isomorphism can be efficiently reduced to this geometric automorphism problem, which yields the $O^*(k^{O(k)})$ time algorithm for it.

2 Preliminaries

Let $[n]$ denote the set $\{1, \dots, n\}$. We assume basic familiarity with the notions of vector spaces, linear transformations and matrices. The projection of a vector $v \in \mathbb{R}^n$ on a subspace $S \subseteq \mathbb{R}^n$ is denoted as $\text{proj}_S(v)$. The *inner product* of vectors $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ is $\langle u, v \rangle = \sum_{i \in [n]} u_i v_i$. The *euclidean*

norm, $\|u\|$, of a vector u , is $\sqrt{\langle u, u \rangle}$, and the *distance* between two points u and v in \mathbb{R}^n is $\|u - v\|$. Vectors u, v are *orthogonal* if $\langle u, v \rangle = 0$. Subspaces U, V are orthogonal if for every $u \in U, v \in V$, u, v are orthogonal. A set of subspaces W_1, \dots, W_r is said to be an *orthogonal decomposition* of \mathbb{R}^n if each pair of subspaces are mutually orthogonal, and they span \mathbb{R}^n . A square matrix M is orthogonal if $M^T M = I$. A linear transformation T *stabilizes* a subspace S if $T(S) \subseteq S$. Given a matrix M , we call λ to be an *eigenvalue* of M if there exists a vector v such that $Mv = \lambda v$. We call v to be an *eigenvector* of M of eigenvalue λ . The set of all eigenvectors of M of eigenvalue λ is a subspace of \mathbb{R}^n . The following well-known fact about $n \times n$ symmetric matrices will be useful.

Fact 1. *All eigenvalues of a symmetric matrix are real. Moreover, the eigenspaces form an orthogonal decomposition of \mathbb{R}^n .*

We use $\text{Sym}(V)$ to denote group of all permutations on a finite set V . Given a graph $X = (V, E)$, a permutation $\pi \in \text{Sym}(V)$ is an *automorphism* of the graph X if for all pairs $\{u, v\}$ of vertices, $\{u, v\} \in E$ iff $\{\pi(u), \pi(v)\} \in E$. In other words, π preserves adjacency in X . The set of all automorphisms of a graph X , denoted by $\text{Aut}(X)$, is a subgroup of $\text{Sym}(V)$, which is denoted by $\text{Aut}(X) \leq \text{Sym}(V)$.

We can similarly talk of automorphisms of hypergraphs: Let $X = (V, E)$ be a hypergraph with vertex set V and edge set $E \subset 2^V$. A permutation $\pi \in \text{Sym}(V)$

is an *automorphism* of the *hypergraph* X if for every subset $e \subseteq V$, $e \in E$ if and only if $\pi(e) \in E$, where $\pi(e) = \{\pi(v) \mid v \in e\}$.

Given an undirected graph $X = (V, E)$, the set V indexed by $[n]$, we define its *adjacency matrix* A_X is defined as follows: $A_X(i, j) = 1$ if $\{v_i, v_j\} \in E$ and 0 otherwise. Clearly, the adjacency matrix A_X of an undirected graph X is symmetric. Given a permutation $\pi : [n] \rightarrow [n]$, we can associate a natural permutation matrix M_π with it. It is easy to verify that π is an automorphism of a graph G iff $M_\pi^T A_X M_\pi = A_X$. Since permutation matrices are orthogonal matrices, the following simple folklore lemma characterizes the automorphisms of a graph through the action of the associated matrix on the eigenspaces of its adjacency matrix.

Lemma 1. *Let X be the adjacency matrix of a graph $G = (V, E)$. Then, a permutation $\pi \in \text{Sym}(V)$ is an automorphism of G iff the associated linear map M_π preserves the eigenspaces of X .*

Proof. Suppose $\pi \in \text{Aut}(G)$. Then $M_\pi A_X = A_X M_\pi$ and therefore, for any eigenvector v in eigenspace W_i of eigenvalue λ_i , $A_X M_\pi v = M_\pi A_X v = \lambda_i M_\pi v$ which shows that $M_\pi v \in W_i$. Conversely, suppose M_π preserves eigenspaces W_i of X . Then, for any $v \in W_i$, $A_X M_\pi v = \lambda_i M_\pi v = M_\pi A_X v$. Since eigenvectors of the symmetric matrix A_X span \mathbb{R}^n , this implies that $A_X M_\pi = M_\pi A_X$. Therefore, π must be an automorphism of G . \square

Remark 1. *Our approach to solving Graph Isomorphism for bounded eigenvalue multiplicity is based on a variation of this lemma, as described in Proposition 2. We first map the graph G into a point set \mathcal{P} in the n -dimensional space \mathbb{R}^n . Then, we project \mathcal{P} into eigenspace W_i of G , to obtain \mathcal{P}_i , for each eigenspace W_i . It turns out that π is an automorphism of G if and only if π , in its induced action is a congruence for the point set \mathcal{P}_i for each eigenspace W_i . When the eigenspaces W_i are of dimension bounded by the parameter k , it creates the setting for application of the $O^*(k^{O(k)})$ -time algorithm for GGI [AR14].*

Next, we recall some useful results about permutation group algorithms. Further details can be found in the excellent text of Ser  ss [Ser].

A *permutation group* is a subgroup $G \leq \text{Sym}(\Omega)$ of the group of all permutations on a finite domain Ω . A subset $A \subseteq G$ of a permutation group G is a *generating set* for G if every element of G can be expressed as a product of elements of A . Every permutation group $G \leq \text{Sym}(\Omega)$ has a generating set of size $\log |G| \leq n \log n$ where $n = |\Omega|$. Thus, algorithmically, a compact input representation for permutation groups is by a generating set of size at most $n \log n$. With this input representation, it turns out there several natural permutation group problems have efficient polynomial-time algorithms. A fundamental problem here is *membership testing*: Given a permutation $\pi \in \text{Sym}(\Omega)$ and permutation group G by a generating set, there is a polynomial-time algorithm (the Schreier-Sims algorithm [Ser]) to check if $\pi \in G$. The *pointwise stabilizer* of a subset $\Delta \in \Omega$ in a permutation group $G \leq \text{Sym}(\Omega)$ is the subgroup

$$G_{\{\Delta\}} = \{\pi \in G \mid \forall \gamma \in \Gamma, \pi(\gamma) = \gamma\}.$$

Given a permutation group $G \leq \text{Sym}(\Omega)$ by a generating set, a generating set for $G_{\{\Delta\}}$ in polynomial time using ideas from the Schreier-Sims algorithm [Ser]. More generally, suppose $G \leq \text{Sym}(\Omega)$ is given by a generating set and $\sigma \in \text{Sym}(\Omega)$ is a permutation. The subset of permutations $(G\sigma)_{\Delta} = \{\pi \in G\sigma \mid \pi(\gamma) = \gamma \forall \gamma \in \Delta\}$ that pointwise fix Δ is a right coset $G_{\{\pi^{-1}(\Delta)\}}\tau$ and a generating set for $G_{\{\pi^{-1}(\Delta)\}}$ and such a coset representative τ can be computed in polynomial time [Ser]. We often use the following group-theoretic fact.

Fact 2. *Let $H_i \leq \text{Sym}(\Omega)$, $1 \leq i \leq t$ and $\sigma_i \in \text{Sym}(\Omega)$, $1 \leq i \leq t$, where each H_i is given by a generating set A_i . Suppose the union of the right cosets $\bigcup_{i=1}^t H_i \sigma_i$ is a coset $G\sigma$ for some subgroup $G \leq \text{Sym}(\Omega)$. Then, we can choose the coset representative σ to be σ_1 and the set $\bigcup_{i=1}^t A_i \cup \{\sigma_i \sigma_1^{-1} \mid 2 \leq i \leq t\}$ is a generating set for G .*

3 Algorithm Overview

Before we give an overview of the main result of this paper, we recall the Point Set Congruence problem (also known as the geometric isomorphism problem) GGI [AMW⁺88, Ak98, BK00].

Given two finite n -point sets A and B in \mathbb{Q}^k , we say A and B are *isomorphic* if there is a *distance-preserving* bijection between A and B , where the distance is in the l_2 metric. The *Geometric Graph Isomorphism* problem, denoted GGI, is to decide if A and B are isomorphic. This problem is also known as *Point Set Congruence* in the computational geometry literature [Ak98, BK00, AMW⁺88]. It is called “Geometric Graph Isomorphism” by Evdokimov and Ponomarenko in [EP97b], which we find more suitable as the problem is closely related to Graph Isomorphism. In [AR14] we obtained a $O^*(k^{O(k)})$ time algorithm for this problem.

We now begin with a definition.

Definition 1. *Let $\mathcal{P} = \{p_1, p_2, \dots, p_m\} \subset \mathbb{Q}^n$ be a finite point set. A geometric automorphism of \mathcal{P} is a permutation π of the point set \mathcal{P} such that for each pair of points $p_i, p_j \in \mathcal{P}$ we have*

$$\begin{aligned} \|p_i\| &= \|\pi(p_i)\|, \text{ and} \\ \|p_i - p_j\| &= \|\pi(p_i) - \pi(p_j)\|, \end{aligned}$$

where p_i denotes, by abuse of notation, also the position vector of the point p_i .

Let $\mathcal{P} = \{p_1, p_2, \dots, p_m\} \subset \mathbb{Q}^n$ be a finite point set such that their set of position vectors $\{p_i\}$ spans \mathbb{R}^n . We refer to \mathcal{P} as a full-dimensional point set in \mathbb{R}^n .

Proposition 1. *Let $\mathcal{P} = \{p_1, p_2, \dots, p_m\} \subset \mathbb{Q}^n$ be a full-dimensional point set. Then there is a unique orthogonal $n \times n$ matrix A_π such that $A_\pi(p_i) = \pi(p_i)$ for each $p_i \in \mathcal{P}$.*

Proof. As \mathcal{P} is full dimensional, we can define a unique matrix A_π by extending π linearly to all of \mathbb{R}^n . A_π can be shown to be orthogonal as follows. Any vector $x \in \mathbb{R}^n$, x is a linear combination $\sum_{i=1}^n \sigma_i v_i$ where $v_i \in \mathcal{P}$. Then, $\|Ax\|^2 = \sum_{i,j} \sigma_i \sigma_j v_i A^T A v_j$. It suffices to observe that $2v_i A^T A v_j = \|A(v_i - v_j)\|^2 - \|Av_i\|^2 - \|Av_j\|^2 = \|v_i - v_j\|^2 - \|v_i\|^2 - \|v_j\|^2 = 2v_i^T v_j$ for any vectors $v_i, v_j \in \mathcal{P}$. \square

The geometric automorphism problem is defined below:

Problem 1 (GEOM-AUT $_k$).

Input: A point set $\{p_1, p_2, \dots, p_m\} \subset \mathbb{Q}^n$ and an orthogonal decomposition of $\mathbb{R}^n = W_1 \oplus W_2 \oplus \dots \oplus W_r$, where $\dim(W_i) \leq k$ and $W_i \perp W_j$ for all $i \neq j$.

Parameter: k .

Output: The subgroup $G \leq S_m$ consisting of all automorphisms π of the input point set such that the orthogonal matrix A_π stabilizes each subspace W_i .

The $O^*(k^{O(k)})$ time algorithm for EVGI $_k$ has the following three steps.

1. We give a polynomial-time reduction from EVGI $_k$ to GEOM-AUT $_{2k}$.
2. We apply the $O^*(k^{O(k)})$ time algorithm for GGI [AR14] to give a $O^*(k^{O(k)})$ time reduction from GEOM-AUT $_{2k}$ to a special hypergraph automorphism problem HYP-AUT.
3. We give a polynomial-time dynamic programming algorithm for HYP-AUT by adapting the hypergraph isomorphism algorithm for bounded color classes in [ADKT10].

Proposition 2. *There is a deterministic polynomial-time reduction from EVGI $_k$ with parameter k to GEOM-AUT $_{2k}$ with parameter $2k$.*

Proof. Let $X = X_1 \cup X_2$ be the disjoint union of the input instance (X_1, X_2) of EVGI $_k$. The adjacency matrix A_X of X is block diagonal and has the adjacency A_{X_1} and A_{X_2} as its two blocks along the diagonal. Thus, A_X has the same set of eigenvalues as A_{X_1} and A_{X_2} , and the multiplicity at most doubles.³ Clearly, we can decide whether X_1 and X_2 are isomorphic by computing $\text{Aut}(X)$ and checking if there exists a $\pi \in \text{Aut}(X)$ such that $\pi(X_1) = X_2$ and vice-versa.

Furthermore, by Lemma 1 a permutation $\pi \in \text{Sym}(V(X))$ is an automorphism of X if and only if π (considered as a linear map on \mathbb{R}^{2n}) preserves each eigenspace of X . Let $\lambda_1, \lambda_2, \dots, \lambda_r$ be the r eigenvalues of X and W_1, W_2, \dots, W_r be the corresponding eigenspaces.⁴

Next, we compute the point set $\mathcal{P} = \{p_1, p_2, \dots, p_{m+2n}\}$ corresponding to the graph $X = (V, E)$, where $|V| = 2n$ and $|E| = m$. The points p_1, p_2, \dots, p_{2n}

³We can assume w.l.o.g. that A_{X_1} and A_{X_2} have the same eigenvalues with the same multiplicity as we can check that in polynomial time.

⁴By applying suitable numerical methods we can compute each λ_i and basis for each W_i to polynomially many bits of accuracy in polynomial time. This suffices for our algorithms.

are defined by the elementary n -dimensional vectors $e_i \in \mathbb{R}^{2n}, 1 \leq i \leq 2n$. The points $p_{2n+1}, \dots, p_{2n+m}$ are defined by vectors corresponding to the edges in E as follows: For each edge $e = \{i, j\} \in E$ the corresponding point has 1 in the i^{th} and j^{th} locations and zeros elsewhere.

We claim that $\pi \in \text{Aut}(X)$ iff π is a geometric automorphism of \mathcal{P} . Let π be any permutation on the vertex set $V(X)$. The action of the permutation π extends (uniquely) to the edge set, and hence to the point set \mathcal{P} as well. If $\pi \in \text{Aut}(X)$ then, clearly, π is a geometric automorphism for the point set \mathcal{P} . Conversely, if π is geometric automorphism of the point set \mathcal{P} then it stabilizes the subset of points $\{p_1, \dots, p_{2n}\}$ encoding vertices and the subset $\{p_{2n+1}, \dots, p_{2n+m}\}$ encoding edges which means $\pi \in \text{Aut}(X)$. This completes the reduction and its correctness proof. \square

4 The Geometric Automorphism Problem GEOM-AUT_k

In this section, we introduce some necessary definitions and state a useful characterization of a geometric isomorphism of a set of points. This will lead to our $O^*(k^{O(k)})$ time algorithm for GEOM-AUT_k which yields the main result for EVGI_k by Proposition 2.

Let $(\mathcal{P}, W_1, W_2, \dots, W_r)$ be the instance of GEOM-AUT_k. W.l.o.g. we can assume that \mathcal{P} is full dimensional. Otherwise, we can cut down the dimensional of the ambient space \mathbb{R}^n to the dimension of the point set \mathcal{P} .

We can assume w.l.o.g. that each W_ℓ is given by a basis $u_{\ell 1}, u_{\ell 2}, \dots, u_{\ell k_\ell}$ where $k_\ell \leq k$ for all $\ell \in [r]$.

Each point $p_i \in \mathcal{P}$ has its projection $\text{proj}_\ell(p_i)$ in the subspace W_ℓ defining the projection $\mathcal{P}_\ell = \text{proj}_\ell(\mathcal{P})$ inside W_ℓ of the point set \mathcal{P} . For each $p_i \in \mathcal{P}$ we can uniquely express it as

$$p_i = \sum_{\ell=1}^r \text{proj}_\ell(p_i).$$

Thus we have the projections $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_r$ of the input point set \mathcal{P} into the orthogonal subspaces W_1, W_2, \dots, W_r , respectively. These projections naturally define equivalence relations on the point set \mathcal{P} as follows.

Definition 2. *Two points $p_i, p_j \in \mathcal{P}$ are (ℓ) -equivalent if $\text{proj}_\ell(p_i) = \text{proj}_\ell(p_j)$, and they are $[\ell]$ -equivalent if $\text{proj}_t(p_i) = \text{proj}_t(p_j), 1 \leq t \leq \ell$.*

Since $\mathbb{R}^n = W_1 \oplus W_2 \oplus \dots \oplus W_r$ we observe the following.

Fact 3. *For any two $p_i, p_j \in \mathcal{P}$ we have $p_i = p_j$ iff p_i and p_j are $[r]$ -equivalent.*

In other words, the common refinement of the (ℓ) -equivalence relations, $1 \leq \ell \leq r$, is the identity relation on \mathcal{P} , and the equivalence classes of this refinement are the singleton sets. Given a permutation π on the point set \mathcal{P} we can ask whether it induces an automorphism on the projection \mathcal{P}_ℓ in the following sense.

A subset $\Delta \subset \mathcal{P}$ of points is an (ℓ) -equivalence class of \mathcal{P} if and only if for some point $p \in \mathcal{P}_\ell$ we have $\Delta = \text{proj}_\ell^{-1}(p)$. Thus, each point in the projected set

\mathcal{P}_ℓ represents an (ℓ) -equivalence class. We say that permutation $\pi \in \text{Sym}(\mathcal{P})$ *respects* \mathcal{P}_ℓ iff for each (ℓ) -equivalence class $\Delta \subset \mathcal{P}$ the subset $\pi(\Delta)$ is an (ℓ) -equivalence class. Suppose $\pi \in \text{Sym}(\mathcal{P})$ is a permutation that respects \mathcal{P}_ℓ . Then π induces a permutation π_ℓ on the point set \mathcal{P}_ℓ as follows: for each $p \in \mathcal{P}_\ell$ its image is

$$\pi_\ell(p) = \text{proj}_\ell(\pi(\text{proj}_\ell^{-1}(p))).$$

Definition 3. A permutation $\pi \in \text{Sym}(\mathcal{P})$ is said to be an induced geometric automorphism on the projection $\mathcal{P}_\ell \subset W_\ell$ if π respects \mathcal{P}_ℓ and π_ℓ is a geometric automorphism of the point set \mathcal{P}_ℓ .

Lemma 2. Let $(\mathcal{P}, W_1, W_2, \dots, W_r)$ be an instance of GEOM-AUT_k and \mathcal{P} be full dimensional in \mathbb{R}^n . Let π be a permutation on \mathcal{P} . Then π is a geometric automorphism of \mathcal{P} such that $A_\pi(W_\ell) = W_\ell$ for each $\ell \in [r]$ if and only if π is an induced automorphism of each $\mathcal{P}_\ell, 1 \leq \ell \leq r$.

Proof. For the forward direction, suppose π is a geometric automorphism of \mathcal{P} such that $A_\pi(W_\ell) = W_\ell$ for each W_ℓ . We claim that π is an induced automorphism of \mathcal{P}_ℓ for each ℓ .

For any point $p_i \in \mathcal{P}$ we can write

$$p_i = \text{proj}_\ell(p_i) + u,$$

where u is a vector in W_ℓ^\perp . Since A_π stabilizes each W_i , it follows by linearity that

$$\text{proj}_\ell(A_\pi(p_i)) = A_\pi(\text{proj}_\ell(p_i)).$$

Hence $A_\pi(\mathcal{P}_\ell) = \mathcal{P}_\ell$ which implies π is an induced automorphism of \mathcal{P}_ℓ for each ℓ .

Conversely, suppose a permutation π on \mathcal{P} is an induced automorphism of each $\mathcal{P}_\ell, 1 \leq \ell \leq r$. Since each \mathcal{P}_ℓ is a full-dimensional point set in W_ℓ , it follows that $A_\pi(W_\ell) = W_\ell$ for each ℓ . To see that π is a geometric automorphism of \mathcal{P} , let $p_i, p_j \in \mathcal{P}$. We can write $p_i = \sum_{\ell=1}^r \text{proj}_\ell(p_i)$ and $p_j = \sum_{\ell=1}^r \text{proj}_\ell(p_j)$. By linearity, we have $A_\pi(p_i) = \sum_{\ell=1}^r A_\pi(\text{proj}_\ell(p_i))$ and $A_\pi(p_j) = \sum_{\ell=1}^r A_\pi(\text{proj}_\ell(p_j))$. Hence, by Pythagoras theorem we have

$$\begin{aligned} \|A_\pi(p_i) - A_\pi(p_j)\|^2 &= \sum_{\ell=1}^r \|A_\pi(\text{proj}_\ell(p_i)) - A_\pi(\text{proj}_\ell(p_j))\|^2 \\ &= \sum_{\ell=1}^r \|\text{proj}_\ell(p_i) - \text{proj}_\ell(p_j)\|^2 \\ &= \|p_i - p_j\|^2, \end{aligned}$$

where the third line above follows because π is an induced automorphism of each \mathcal{P}_ℓ . \square

5 The Hypergraph Automorphism Problem

By Lemma 2 it follows that $\text{Aut}(\mathcal{P})$ is the group of all $\pi \in \text{Sym}(\mathcal{P})$ such that π is an induced automorphism of each $\mathcal{P}_\ell, 1 \leq \ell \leq r$. In this section we describe the algorithm for computing a generating set for $\text{Aut}(\mathcal{P})$ in $O^*(k^{O(k)})$ time.

The first step is to reduce GEOM-AUT_k in $O^*(k^{O(k)})$ time to a hypergraph automorphism problem defined below:

Problem 2 (HYP-AUT).

Input: A hypergraph $X = (V, E)$ and a partition of the vertex set into color classes $V = V_1 \cup V_2 \cup \dots \cup V_r$, and subgroups $G_i \leq \text{Sym}(V_i), 1 \leq i \leq r$, where each G_i is given as an explicit list of permutations.

Output: A generating set for $\text{Aut}(X) \cap G_1 \times G_2 \times \dots \times G_r$.

We will give a polynomial-time algorithm for this problem based on a dynamic programming strategy as used in [ADKT10]. Before that we will show that GEOM-AUT_k is reducible to HYP-AUT in $O^*(k^{O(k)})$ time. Combining the two we will obtain the $O^*(k^{O(k)})$ time algorithm for GEOM-AUT_k .

Theorem 1. *There is a $O^*(k^{O(k)})$ time reduction from the GEOM-AUT_k problem to HYP-AUT.*

Proof. Let $(\mathcal{P}, W_1, W_2, \dots, W_r)$ be an instance of GEOM-AUT_k . In order to compute $\text{Aut}(\mathcal{P})$ we first compute each $\mathcal{P}_\ell, \ell \in [r]$. Then, since W_ℓ is k -dimensional we can compute the geometric automorphisms $\text{Aut}(\mathcal{P}_\ell)$ in $O^*(k^{O(k)})$ time by applying the main result of [AR14]. Indeed, $\text{Aut}(\mathcal{P}_\ell)$ can be explicitly listed down in $O^*(k^{O(k)})$ time, also implying that $|\text{Aut}(\mathcal{P}_\ell)|$ is bounded by $O^*(k^{O(k)})$. Now, we construct a hypergraph instance $X = (V, E)$ of HYP-AUT as follows: The vertex set V is the disjoint union $V = \mathcal{P}_1 \cup \dots \cup \mathcal{P}_r$, and the explicitly listed groups $G_\ell = \text{Aut}(\mathcal{P}_\ell), \ell \in [r]$. For each point $p_i \in \mathcal{P}$ we include a hyperedge $e_p \in E$, where $e_p = \{\text{proj}_1(p_i), \text{proj}_2(p_i), \dots, \text{proj}_r(p_i)\}$. Since the edges of X encode points in \mathcal{P} , the induced action of the automorphism group $\text{Aut}(X) \cap G_1 \times G_2 \times \dots \times G_r$ on the edges of X is in one-to-one correspondence with $\text{Aut}(\mathcal{P})$ by Lemma 2. Hence, we can obtain a generating set for $\text{Aut}(\mathcal{P})$. Clearly, the reduction runs in time $O^*(k^{O(k)})$. \square

In the polynomial-time algorithm for HYP-AUT we will use as subroutine a polynomial-time algorithm for the following simple coset intersection problem.

Problem 3 (Restricted Coset Intersection).

Input: Let $V = V_1 \uplus V_2 \uplus \dots \uplus V_r$ be a partition of the domain into color classes and $G_i \leq \text{Sym}(V_i)$ be an explicitly listed subgroup of permutations on $V_i, 1 \leq i \leq r$. Let H and H' be subgroups of the product group $G_1 \times \dots \times G_r$, where H and H' are given by generating sets as input. Let $\pi, \pi' \in G_1 \times \dots \times G_r$.

Output: The coset intersection $H\pi \cap H'\pi'$ which, if nonempty, is given by a generating set for $H \cap H'$ and a coset representative $\pi'' \in H\pi \cap H'\pi'$.

Lemma 3. *The above restricted coset intersection problem has a polynomial-time algorithm.*

Proof. We give a sketch of the algorithm which is a simple application of the classical Schreier-Sims algorithm (mentioned in Section 2): given a permutation group $G \leq \text{Sym}(\Omega)$ by a generating set and another permutation $\pi \in \text{Sym}(\Omega)$, for any point $\alpha \in \Omega$ the subcoset of $G\pi$ that fixes the point α can be computed in time polynomial in $|\Omega|$ and the size of the generating set for G . See, e.g. [Ser] for details.

In order to compute the intersection $H\pi \cap H'\pi'$, we consider the product group $H \times H'$ acting on the set $\Delta = \bigcup_{i=1}^r V_i \times V_i$ component-wise. The permutation pair (π, π') too defines a permutation on the set Δ . We consider now the coset $(H \times H')(\pi, \pi')$ of the group $H \times H'$. Define the diagonal sets

$$D_i = \{(\alpha, \alpha) \mid \alpha \in V_i\}, 1 \leq i \leq r.$$

The following claim is immediate from the definitions.

Claim 1. *A pair $(h, h') \in (H \times H')(\pi, \pi')$ maps each D_i to D_i if and only if $h = h'$ and $h \in H\pi \cap H'\pi'$.*

Thus, in order to compute the coset intersection it suffices to compute the subcoset

$$\{(h, h') \in (H \times H')(\pi, \pi') \mid (h, h')(D_i) = (D_i) 1 \leq i \leq r\}$$

of the coset $(H \times H')(\pi, \pi')$. Notice that $D_i \subset V_i \times V_i$ and the elements of the coset $(H \times H')(\pi, \pi')$ restricted to $V_i \times V_i$ are from the group $G_i \times G_i$ which is polynomially bounded in input size. Let Ω denote the entire orbit of D_i under the action of the group $G_i \times G_i$. Clearly, $|\Omega| \leq |G_i|^2$ and therefore is polynomially bounded in input size and can be computed. Now, D_i is just a point in the set Ω and we can compute its pointwise stabilizer subcoset in $(H \times H')(\pi, \pi')$ by the Schreier-Sims algorithm (as outlined above) in time polynomial in $|\Omega|$ and the generating sets sizes of H and H' . Repeating this procedure for each $D_i, 1 \leq i \leq r$ yields the subcoset that maps D_i to D_i for each i . This completes the proof sketch. \square

We now describe the polynomial-time algorithm for HYP-AUT.

Theorem 2. *There is a polynomial-time algorithm for HYP-AUT.*

Proof. The algorithm is a dynamic programming strategy exactly as in [ADKT10]. But, unlike the problem considered in [ADKT10], we do not have bounded-size color classes in our hypergraph instances. Instead, we have color classes V_i and explicitly listed subgroups $G_i \leq \text{Sym}(V_i)$ on each color class and we have to compute color-class preserving automorphisms $\pi \in \text{Aut}(X)$ that, when restricted to each color class V_i belong to the corresponding G_i . We now describe the algorithm.

The subproblems of this dynamic programming algorithm involve hypergraphs (V, E) with multiple hyperedges (i.e., E is a multi-set). Thus, we may assume that the input X too is a *multi-hypergraph* given with the vertex set partition $V = \uplus_{\ell=1}^r V_\ell$, and groups $G_\ell \leq \text{Sym}(V_\ell)$ explicitly listed as permutations. A bijection $\varphi : V \rightarrow V$ is an automorphism of interest if φ maps each V_ℓ to V_ℓ such that:

- The permutation φ restricted to V_ℓ is an element of the group G_ℓ .
- The map induced by φ on E preserves the hyperedges with their multiplicities (for each hyperedge $e \subseteq V$, e and $\varphi(e)$ have the same multiplicity in E).

We first introduce some notation. For $\ell \in [r]$ and any multi-set D of hyperedges $e \subseteq V$, let $D_{[\ell]}$ denote the multi-hypergraph $(V_{[\ell]}, \{e \cap V_{[\ell]} \mid e \in D\})$ on vertex set $V_{[\ell]} = V_1 \uplus \dots \uplus V_\ell$. Further, let D_ℓ denote the multi-hypergraph $(V_\ell, \{e \cap V_\ell \mid e \in D\})$ on vertex set V_ℓ . For two multi-hypergraphs $D_{[\ell]}$ and $D'_{[\ell]}$ let $\text{ISO}(D_{[\ell]}, D'_{[\ell]})$ denote the coset of all isomorphisms between them that belong to $G_1 \times \dots \times G_\ell$.

For $\ell \in [r]$ we define an equivalence relation \equiv_ℓ on the hyperedges in E : for hyperedges $e_1, e_2 \in E$ we say $e_1 \equiv_\ell e_2$ if

$$e_1 \cap V_j = e_2 \cap V_j \text{ for } j = \ell + 1, \dots, r.$$

The equivalence classes of \equiv_ℓ are called (ℓ) -blocks. For $\ell \leq j$, notice that \equiv_ℓ is a refinement of \equiv_j . Thus, if e_1 and e_2 are in the same (ℓ) -block then they are in the same (j) -block for all $j \geq \ell$.

The algorithm works in stages $\ell = 0, \dots, r$. In stage ℓ , the algorithm considers the multi-hypergraphs $A_{[\ell+1]}$ induced by the different (ℓ) -blocks A on the vertex set $V_{[\ell+1]}$. For each pair of (ℓ) -blocks A, B the algorithm computes the cosets $\text{ISO}(A_{[\ell]}, B_{[\ell]})$ (unless $\ell = 0$) using the cosets of the form $\text{ISO}(A_{[\ell-1]}^i, B_{[\ell-1]}^j)$ computed already. Finally, for the single (r) -block E the algorithm computes the coset $\text{ISO}(E_{[r]}, E_{[r]})$ which is the desired group $\text{Aut}(X) \cap G_1 \times \dots \times G_r$.

Stage 0: Let A and B be (0) -blocks. Then A contains a single hyperedge a with multiplicity $|A|$, and B contains b with multiplicity $|B|$. The coset $\text{ISO}(A_{[1]}, B_{[1]}) = \emptyset$ if $\|A\| \neq \|B\|$ or $\|a \cap V_1\| \neq \|b \cap V_1\|$. Otherwise, $\text{ISO}(A_{[1]}, B_{[1]}) \cap G_1$ is a subcoset of all elements of G_1 that maps $a \cap V_1$ to $b \cap V_1$, which can be computed by inspecting the list of elements in G_1 .

For $\ell := 1$ to $r - 1$ do

Stages ℓ : For each pair (A, B) of (ℓ) -blocks compute the table entry $T(\ell, A, B) = \text{ISO}(A_{[\ell]}, B_{[\ell]})$ as follows:

1. Partition the (ℓ) -blocks A and B into $(\ell - 1)$ -blocks A^1, \dots, A^t and $B^1, \dots, B^{t'}$, respectively. If $t \neq t'$ then $\text{ISO}(A_{[\ell]}, B_{[\ell]})$ is empty.
2. Otherwise, $t = t'$. Clearly, for all $e \in A^i$, $e \cap V_\ell$ is identical. Let $a_i = e \cap V_\ell$, $e \in A^i$ and $b_{i'} = e \cap V_\ell$, $e \in B^{i'}$, for $1 \leq i, i' \leq t$. Let $S_\ell \subset G_\ell$ be the subcoset of all permutations $\tau \in G_\ell$ such that τ (injectively) maps the set $\{a_1, a_2, \dots, a_t\}$ to the set $\{b_1, b_2, \dots, b_t\}$. For each $\tau \in S_{|ell|}$, we denote by $\hat{\tau}$ this induced mapping that injectively maps the set $\{a_i \mid 1 \leq i \leq t\}$ to $\{b_{\hat{\tau}(i)} \mid 1 \leq i \leq t\}$.

We can compute S_ℓ in polynomial time since G_ℓ is given as an explicit list as part of the input.

3. For $\tau \in S_\ell$, recall that $A_{[\ell-1]}^j$ and $B_{[\ell-1]}^{\hat{\tau}(j)}$ denote the multi-hypergraphs obtained from the $(\ell - 1)$ -blocks A^j and $B^{\hat{\tau}(j)}$, where $j \mapsto \hat{\tau}(j)$ for $\tau \in S_\ell$ means that τ maps a_j to $b_{\tau(j)}$. Then it is clear that we have

$$\text{ISO}(A_{[\ell]}, B_{[\ell]}) = \bigcup_{\tau \in S_\ell} \bigcap_{j=1}^t \text{ISO}(A_{[\ell-1]}^j, B_{[\ell-1]}^{\hat{\tau}(j)}) \times \{\tau\} \quad (1)$$

where we have already computed the coset $\text{ISO}(A_{[\ell-1]}^j, B_{[\ell-1]}^{\pi(j)})$.

4. In order to compute the coset $\text{ISO}(A_{[\ell]}, B_{[\ell]})$ from Equation 1, we cycle through the polynomially many $\tau \in S_\ell$, and compute each coset intersection $\bigcap_{j=1}^t \text{ISO}(A_{[\ell-1]}^j, B_{[\ell-1]}^{\hat{\tau}(j)})$ by repeated application of the restricted coset intersection algorithm of Lemma 3. We can write a generating set for the union of the cosets over all τ using Fact 2.

Output: In the last step, the unique (r) -block is the entire set of hyperedges E , and the table entry $T(r, E_{[r]}, E_{[r]}) = \text{ISO}(E_{[r]}, E_{[r]})$.

It is clear from the description that the running time is polynomially bounded in $|E|, |V|$ and $\max_{1 \leq \ell \leq r} |G_\ell|$. \square

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